# NAGATA-ASSOUAD DIMENSION VIA LIPSCHITZ EXTENSIONS

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ABSTRACT. In the first part of the paper we show how to relate several dimension theories (asymptotic dimension with Higson property, asymptotic dimension of Gromov, and capacity dimension of Buyalo [7]) to Nagata-Assouad dimension. This is done by applying two functors on the Lipschitz category of metric spaces: microscopic and macroscopic. In the second part we identify (among spaces of finite Nagata-Assouad dimension) spaces of Nagata-Assouad dimension at most $n$ as those for which the $n$-sphere $S^n$ is a Lipschitz extensor. Large scale and small scale analogs of that result are given.

## CONTENTS

## 1. INTRODUCTION

The large-scale geometry of metric spaces has been the subject of intense research in the last 15 years. For a comprehensive account of the area see Gromov's paper [12]. There are two large-scale concepts related to the topic of our paper: *asymptotic dimension* and *asymptotic dimension of linear type* (Dranishnikov and Zarichnyi [11] refer to the latter as *asymptotic dimension with Higson property*). Asymptotic dimension is an invariant of the coarse

category of Roe [19]. Asymptotic dimension of linear type is preserved by bi-Lipschitz functions, so its natural place is in the Lipschitz category.

Notice that our notion of a Lipschitz function is a bit different from that used in [13]. Namely, we allow Lipschitz constant to be smaller than 1. Thus, a Lipschitz function $f : (X, d_X) \to (Y, d_Y)$ satisfies $d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y)$ for some $\lambda \geq 0$ and all $x, y \in X$. The infimum of all possible $\lambda$ is denoted by $Lip(f)$. $f$ is called *bi-Lipschitz* if there are constants $\mu, \lambda > 0$ such that $\mu \cdot d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y)$ for all $x, y \in X$.

In [14] a variation of asymptotic dimension is considered. That invariant of bi-Lipschitz functions was introduced and named *Nagata dimension* by Assouad [1] as it is closely related to a theorem of Nagata characterizing the topological dimension of metrizable spaces (cf. [[16], Thm. 5] or [[17], p. 138]). In our paper we refer to it as *Nagata-Assouad dimension*. In contrast to the asymptotic dimension, the Nagata dimension of a metric space is in general not preserved under quasi-isometries, but it is still a bi-Lipschitz invariant and, as it turns out, even a quasisymmetry invariant (see [14]). The class of metric spaces with finite Nagata dimension includes all doubling spaces, metric trees, euclidean buildings, and homogeneous or pinched negatively curved Hadamard manifolds as shown in [14]. One of main results of [14] relates to theorems of Assouad [2] and Dranishnikov [10]. Namely, Theorem 1.3 of [14] states: Every metric space with Nagata-Assouad dimension at most $n$ admits a quasisymmetric embedding into the product of $n + 1$ metric trees.

The result from [14] of major importance to us is Theorem 1.4:

**Theorem 1.1.** *Suppose that $X$, $Y$ are metric spaces, $dim_{NA} X \leq n < \infty$, and $Y$ is complete. If $Y$ is Lipschitz $m$-connected for $m = 0, 1, \ldots, n-1$, then the pair $(X, Y)$ has the Lipschitz extension property.*

Recall that $(X, Y)$ has the *Lipschitz extension property* if there is a constant $C > 0$ such that for any Lipschitz map $f : A \to Y$, $A$ any subset of $X$, there is a Lipschitz extension $g : X \to Y$ of $f$ such that $Lip(g) \leq C \cdot Lip(f)$. We call $Y$ a *Lipschitz extensor of $X$* in such a case. $Y$ is *Lipschitz $m$-connected* if there is a constant $C_m > 0$ such that any Lipschitz function $f : S^m \to Y$ extends over the $(m + 1)$-ball $B^{m+1}$ to $g : B^{m+1} \to Y$ so that $Lip(g) \leq C_m \cdot Lip(f)$.

One of the main themes of our paper is characterizing Nagata-Assouad dimension via Lipschitz extensions. In Section 4 we characterize spaces of Nagata-Assouad dimension at most $n$ (among all spaces of finite Nagata-Assouad dimension) as those for which $n$-sphere $S^n$ is a Lipschitz extensor. In the case of dimension zero the assumption of $\dim_{NA}(X)$ being finite can be dropped.

Recently, S.Buyalo [7] introduced the *capacity dimension* of a metric space and proved many analogs of results obtained by U. Lang and T. Schlichenmaier [14] for Nagata-Assouad dimension. It is clear that capacity dimension is the small scale version of Nagata-Assouad dimension. In Section 2

we formalize that observation by introducing microscopic and macroscopic functors on the Lipschitz category. That way many results from Section 3 of [7] can be deduced formally from [14]. Also, the main result of [7] (the asymptotic dimension of a visual hyperbolic space $X$ is bounded by 1 plus the capacity dimension of the visual boundary of $X$) is really about Nagata-Assouad dimension of the visual boundary of $X$ - see 2.12.

## 2. Microscopic and macroscopic Nagata-Assouad dimensions

**Definition 2.1.** A metric space $X$ is said to be of *Nagata-Assouad dimension* at most $n$ (notation: $\dim_{NA}(X) \leq n$) if there is $C > 0$ such that for all $r > 0$ there is a cover $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint and the diameter of elements of $\mathcal{U}$ is bounded by $C \cdot r$.

Nagata-Assouad dimension can be characterized in many ways (see [14]):

**Proposition 2.2.** *For a metric space $(X, d)$ the following conditions are equivalent:*

  (1) *$dim_{NA}(X) \leq n$.*
  (2) *There is a constant $C_1 > 0$ such that for any $r > 0$ there is a cover $\mathcal{U}_r$ of $X$ of multiplicity at most $n + 1$, of mesh at most $C_1 \cdot r$, and of Lebesque number at least $r$.*
  (3) *There is a constant $C_2 > 0$ such that for any $r > 0$ there is a cover $\mathcal{V}_r$ of mesh at most $C_2 \cdot r$ such that each $r$-ball $B(x, r)$ intersects at most $n + 1$ elements of $\mathcal{V}_r$.*

If one replaces all $r > 0$ in 2.1 by $r$ sufficiently small, then one gets the concept of *capacity dimension* of Buyalo [7] for which he proved analog of 2.2. If one replaces all $r > 0$ in 2.1 by $r$ sufficiently large, then one gets the concept of *asymptotic dimension of linear type* or *asymptotic dimension with Higson property*. In this section we will show how to introduce those dimensions formally from Nagata-Assouad dimension. That way small scale version and large scale version of 2.2 are in fact consequences of 2.2.

Given a metric space $(X, d)$ and $\epsilon > 0$ we consider the metric $\max(d, \epsilon)$ on $X$. Needles to say, the formula should not be read literally, only in the case of $x \neq y$. Similarly, we consider the metric $\min(d, \epsilon)$.

A metric space $(X, d)$ is called *discrete* if $(X, d)$ is $\delta$-*discrete* for some $\delta > 0$. That means $d(x, y) > \delta$ for all $x \neq y$.

**Lemma 2.3.** *Any discrete metric space $(X, d)$ is bi-Lipschitz equivalent to $(X, \max(d, \epsilon))$ for all $\epsilon > 0$.*

**Proof.** Suppose $(X, d)$ is $\delta$-discrete. Notice the identity map $id : (X, \max(d, \epsilon)) \to (X, d)$ is 1-Lipschitz and its inverse is $(1 + \frac{\epsilon}{\delta})$-Lipschitz as $d(x, y) \leq \max(d(x, y), \epsilon)) \leq (1 + \frac{\epsilon}{\delta}) \cdot d(x, y)$ for all $x \neq y \in X$. ∎

**Corollary 2.4.** *For any metric space $(X, d)$ and $\epsilon, \delta > 0$ the space $(X, \max(d, \epsilon))$ is bi-Lipschitz equivalent to $(X, \max(d, \delta))$.*

**Proof.** Assume $\epsilon > \delta$ and notice $\max(d_\delta, \epsilon) = d_\epsilon$, where $d_a = \max(d, a)$. Use 2.3. ∎

Since Nagata-Assouad dimension is an invariant of the Lipschitz category (see [14] for a stronger result for quasisymmetric embeddings), one gets the following.

**Corollary 2.5.** *For any metric space $(X, d)$ the Nagata-Assouad dimension of $(X, \max(d, \epsilon))$ does not depend on $\epsilon > 0$.*

Given a metric space $(X, d)$ one can disregard its microscopic features by considering the space $(X, \max(d, 1))$. The *macroscopic Nagata-Assouad dimension* of $(X, d)$ is defined as $\dim_{NA}(X, \max(d, 1))$.

**Corollary 2.6.** *If $(X, d)$ is a discrete metric space, then its macroscopic Nagata-Assouad dimension equals the Nagata-Assouad dimension $\dim_{NA}(X, d)$ of $(X, d)$.*

**Lemma 2.7.** *The macroscopic Nagata-Assouad dimension of a metric space $X$ is at most $n \geq 0$ if and only if there is $C > 0$ such that for sufficiently large $r > 0$ there is a cover $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint and the diameter of elements of $\mathcal{U}$ is bounded by $C \cdot r$.*

**Proof.** Suppose $X$ has a constant $C > 0$ such that for all $r > M$, where $M > 0$, there is a cover $\mathcal{U}_r = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint and the diameter of elements of $\mathcal{U}_r$ is bounded by $C \cdot r$. Let $d_M = \max(d, M)$. Notice the cover $\mathcal{U}_r$, $r \leq M$, consisting of one-point sets is $r$-disjoint in $(X, d_M)$. Since covers $\mathcal{U}_r$, $r > M$, have the same desired properties in $(X, d_M)$ as they do in $(X, d)$, $\dim_{NA}(X, d_M) \leq n$.

If $\dim_{NA}(X, d_1) \leq n$, then for each $r > 1$ there is a a cover $\mathcal{U}_r = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint (in $(X, d_1)$) and the diameter of elements of $\mathcal{U}_r$ is bounded by $C \cdot r$ in $(X, d_1)$. Notice that $\mathcal{U}_i$ is also $r$-disjoint in $(X, d)$ and the diameter of elements of $\mathcal{U}_r$ is bounded by $(C + 1) \cdot r$ in $(X, d)$. ∎

In view of definition of the Higson property in [9] one has the following consequence of 2.7.

**Corollary 2.8.** *If $(X, d)$ is a metric space, then the macroscopic Nagata-Assouad dimension of $(X, d)$ is at most $n$ if and only if $asdim(X) \leq n$ with the Higson property.*

In the reminder of this section we will dualize the above results from large scale/macroscopic category to small scale/microscopic category.

**Lemma 2.9.** *Any bounded metric space $(X, d)$ is bi-Lipschitz equivalent to $(X, \min(d, \epsilon))$ for all $\epsilon > 0$.*

**Proof.** Suppose $(X, d)$ is $\delta$-bounded. Notice the identity map $id : (X, \min(d, \epsilon)) \to (X, d)$ is $(1 + \frac{\epsilon}{\delta})$-Lipschitz and its inverse is 1-Lipschitz as $\min(d(x, y), \epsilon) \leq d(x, y) \leq (1 + \frac{\delta}{\epsilon}) \cdot \min(d(x, y), \epsilon))$ for all $x \neq y \in X$. ∎

**Corollary 2.10.** *For any metric space $(X, d)$ and $\epsilon, \delta > 0$ the space $(X, \min(d, \epsilon))$ is bi-Lipschitz equivalent to $(X, \min(d, \delta))$.*

**Proof.** Assume $\epsilon < \delta$ and notice $\min(d_\delta, \epsilon) = d_\epsilon$, where $d_a = \min(d, a)$. Use 2.9. ∎

Since Nagata-Assouad dimension is an invariant of the Lipschitz category, one gets the following.

**Corollary 2.11.** *For any metric space $(X, d)$ the Nagata-Assouad dimension of $(X, \min(d, \epsilon))$ does not depend on $\epsilon > 0$.*

Given a metric space $(X, d)$ one can disregard its macroscopic features by considering the space $(X, \min(d, 1))$. The *microscopic Nagata-Assouad dimension* of $(X, d)$ is defined as $\dim_{NA}(X, \min(d, 1))$.

**Corollary 2.12.** *If $(X, d)$ is a bounded metric space, then its microscopic Nagata-Assouad dimension equals the Nagata-Assouad dimension $\dim_{NA}(X, d)$ of $(X, d)$.*

**Lemma 2.13.** *The microscopic Nagata-Assouad dimension of a metric space $X$ is at most $n$ if and only if there is $C > 0$ such that for sufficiently small $r > 0$ there is a cover $\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint and the diameter of elements of $\mathcal{U}$ is bounded by $C \cdot r$.*

**Proof.** Suppose $X$ has a constant $C > 0$ such that for all $r < M$, where $M > 0$, there is a cover $\mathcal{U}_r = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint and the diameter of elements of $\mathcal{U}_r$ is bounded by $C \cdot r$. Let $d_M = \min(d, M)$. Notice the cover $\mathcal{U}_r$, $r \geq M$, consisting of the whole $X$ has diameter at most $(C + 1) \cdot r$ in $(X, d_M)$. Since covers $\mathcal{U}_r$, $r < M$, have the same desired properties in $(X, d_M)$ as they do in $(X, d)$, $\dim_{NA}(X, d_M) \leq n$.

If $\dim_{NA}(X, d_1) \leq n$, then for each $r < 1$ there is a a cover $\mathcal{U}_r = \bigcup_{i=1}^{n+1} \mathcal{U}_i$ of $X$ so that each $\mathcal{U}_i$ is $r$-disjoint (in $(X, d_1)$) and the diameter of elements of $\mathcal{U}_r$ is bounded by $C \cdot r$ in $(X, d_1)$. Notice that $\mathcal{U}_i$ is also $r$-disjoint in $(X, d)$ and the diameter of elements of $\mathcal{U}_r$ is bounded by $C \cdot r$ in $(X, d)$. ∎

Since the capacity dimension of Buyalo [7] can be characterized by the condition appearing in 2.13, one derives the following.

**Corollary 2.14.** *If $(X, d)$ is a metric space, then the microscopic Nagata-Assoud dimension of $(X, d)$ is at most $n$ if and only if the capacity dimension of $X$ is at most $n$.*

In [14] it is shown that if $X = A \cup B$, then
$$\dim_{NA}(X) = \max(\dim_{NA}(A), \dim_{NA}(B)).$$

**Corollary 2.15.** *Let $D(Y)$ stand for either the capacity dimension of $Y$ or the asymptotic dimension of linear type of $Y$. If $X = A \cup B$, then*
$$D(X) = \max(D(A), D(B)).$$

**Proof.** Switch to either $\max(d, 1)$ or $\min(d, 1)$ metrics. ∎

## 3. Spheres as Lipschitz extensors

A metric space $E$ is a *Lipschitz extensor* of $X$ if there is a constant $C > 0$ such that any $\lambda$-Lipschitz function $f : A \to E$, $A$ a subset of $X$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to E$.

The purpose of this section is to find necessary and sufficient conditions for a sphere $S^m$ to be a Lipschitz extensor of $X$. This is done by comparing existence of Lipschitz extensions in a finite range of Lipschitz constants to existence of Lebesque refinements in a finite range of Lebesque constants (see 3.1, 3.3, and 3.5).

Given a cover $\mathcal{U} = \{U_s\}_{s \in S}$ of a metric space $(X, d)$ there is a natural family of functions $\{f_s\}_{s \in S}$ associated to $\mathcal{U}$: $f_s(x) := \operatorname{dist}(x, X \setminus U_s)$. To simplify matters by *the local Lebesque number* $L_{\mathcal{U}}(x)$ of $\mathcal{U}$ at $x$ we mean

$$\sup\{f_s(x) \mid s \in S\}$$

and by the (global) *Lebesque number* $L(\mathcal{U})$ of $\mathcal{U}$ we mean

$$\inf\{L_{\mathcal{U}}(x) \mid x \in X\}.$$

We are interested in covers with positive Lebesque number. For those *the local multiplicity* $m_{\mathcal{U}}(x)$ can be defined as $1 + |T(x)|$, where $T(x) = \{s \in S \mid f_s(x) > 0\}$ and the global multiplicity $m(\mathcal{U})$ can be defined as

$$\sup\{m_{\mathcal{U}}(x) \mid x \in X\}.$$

If the multiplicity $m(\mathcal{U})$ is finite, then $\mathcal{U}$ has a natural partition of unity $\{\phi_s\}_{s \in S}$ associated to it:

$$\phi_s(x) = \frac{f_s(x)}{\sum\limits_{t \in S} f_t(x)}.$$

That partition can be considered as a *barycentric map* $\phi : X \to \mathcal{N}(\mathcal{U})$ from $X$ to the *nerve* of $\mathcal{U}$. Since each $f_s$ is 1-Lipschitz, $\sum\limits_{t \in S} f_t(x)$ is $2m(\mathcal{U})$-Lipschitz and each $\phi_s$ is $\frac{2m(\mathcal{U})}{L(\mathcal{U})}$-Lipschitz (use the fact that $\frac{u}{u+v}$ is $\frac{\max(Lip(u), Lip(v))}{\inf(u+v)}$-Lipschitz). Therefore $\phi : X \to \mathcal{N}(\mathcal{U})$ is $\frac{4m(\mathcal{U})^2}{L(\mathcal{U})}$-Lipschitz. See [3] and [8] for more details and better estimates of Lipschitz constants.

Most estimates in this paper work well for both the $l_1$ and $l_2$ metrics on $R^n$ and simplicial complexes.

In analogy to $\lambda$-Lipschitz functions we introduce the concept of *$r$-Lebesque cover* $\mathcal{U}$. That is simply a shortcut to $r \leq L(\mathcal{U})$.

**Proposition 3.1.** *Suppose $X$ is metric, $m \geq 0$, $C > 0$, and $\lambda_2 > \lambda_1 > 0$. If any $\lambda$-Lipschitz function $f : A \to S^m$, $A$ a subset of $X$ and $\lambda_1 < \lambda < \lambda_2$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to S^m$, then $t = \frac{1}{4C(m+2)^2(m+1)}$ has the property that any finite $r$-Lebesque cover $\mathcal{U} = \{U_0, \dots, U_{m+1}\}$ of $X$*

admits a refinement $\mathcal{V}$ so that $\mathcal{V}$ is $t \cdot r$-Lebesque and the multiplicity of $\mathcal{V}$ is at most $m + 1$ provided $\frac{4(m+2)^2}{\lambda_2} < r < \frac{4(m+2)^2}{\lambda_1}$.

**Proof.** Assume $\frac{4(m+2)^2}{\lambda_2} < r < \frac{4(m+2)^2}{\lambda_1}$ and $\mathcal{U}$ is $r$-Lebesque. Therefore $\lambda_1 < \frac{4(m+2)^2}{r} < \lambda_2$. Consider a barycentric map $\phi : X \to \mathcal{N}(\mathcal{U}) = \Delta^{m+1}$. Notice $Lip(\phi) \leq \frac{4(m+2)^2}{r}$. There is $g : X \to \partial\Delta^{m+1}$ such that $Lip(g) \leq \frac{4C(m+2)^2}{r}$ and $g(x) = \phi(x)$ for all $x \in X$ so that $\phi(x) \in \partial\Delta^{m+1}$. Consider $V_i = \{x \in X \mid g_i(x) > 0\}$. Notice $\mathcal{V} = \{V_i\}_{i=0}^{i=m+1}$ is of multiplicity at most $m+1$. Also $x \in V_i$ implies $x \in U_i$, so $\mathcal{V}$ refines $\mathcal{U}$. Given $x \in X$ there is $i$ such that $g_i(x) \geq \frac{1}{m+1}$. If $d(x, y) < \frac{r}{4C(m+2)^2(m+1)}$, then $|g_i(x) - g_i(y)| < \frac{1}{m+1}$ and $g_i(y) > 0$. Thus, the ball at $x$ of radius $\frac{r}{4C(m+2)^2(m+1)}$ is contained in one element of $\mathcal{V}$. That proves $\mathcal{V}$ is $t \cdot r$-Lebesque, where $t = \frac{1}{4C(m+2)^2(m+1)}$. ■

In this paper we consider the space $\mathbb{R}^n$ with either the $l_1$-metric $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ or the $l_2$-metric $d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. The inequalities

$$d_1(x, y) \leq n \cdot \max\{|x_i - y_i|\} \leq n \cdot d_2(x, y)$$

$$d_2(x, y) \leq \sqrt{n} \cdot \max\{|x_i - y_i|\} \leq \sqrt{n} \cdot d_1(x, y)$$

show that the identity map $(\mathbb{R}^n, d_2) \to (\mathbb{R}^n, d_1)$ is $n$-Lipschitz and its inverse is $\sqrt{n}$-Lipschitz.

**Lemma 3.2.** *Let $\triangle$ be a closed convex subset of the space $(\mathbb{R}^n, d_j)$, $j = 1$ or $2$. For any metric space $X$ any $\lambda$-Lipschitz map $f : A \to \triangle$ of a subspace $A \subset X$ can be extended to a $n^2 \cdot \lambda$-Lipschitz map $\tilde{f} : X \to \triangle$.*

**Proof.** Fix an orthogonal coordinate system in $(\mathbb{R}^n, d_2)$. Given a $\lambda$-Lipschitz map $f : A \to (\mathbb{R}^n, d_2)$, every coordinate map $f_i : A \to \mathbb{R}$ is $\lambda$-Lipschitz and can be extended to a $\lambda$-Lipschitz map $\hat{f}_i : X \to \mathbb{R}$ [20]. These coordinate extensions define the map $\hat{f} : X \to \mathbb{R}^n$ which is $\lambda\sqrt{n}$-Lipschitz. Clearly, the nearest point retraction $r_\triangle : (\mathbb{R}^n, d_2) \to \triangle$ is 1-Lipschitz. Therefore the composition $\tilde{f} = r_\triangle \circ \hat{f}$ is $\sqrt{n}\lambda$-Lipschitz.

Since the nearest point retraction $r_\triangle : (\mathbb{R}^n, d_1) \to \triangle$ may be multivalued, we proceed as follows. The composition $id \circ f : A \to (\triangle, d_1) \to (\triangle, d_2)$ is $\sqrt{n}\lambda$-Lipschitz and admits a $n\lambda$-Lipschitz extension $\tilde{f}_2 : X \to (\triangle, d_2)$ by the first part of the proof. Then the composition $\tilde{f} = id \circ \tilde{f}_2 : X \to (\triangle, d_2) \to (\triangle, d_1)$ is $n^2\lambda$-Lipschitz. ■

The idea behind the proof of the next proposition is best understood if one thinks of maps from $X$ to an $(m + 1)$-simplex $\Delta^{m+1}$ as a partition of unity. Since we want to create a map to its boundary $S^m = \partial\Delta^{m+1}$, a geometrical tool is the radial projection $r$ which we splice in the form of $(1 - \beta) \cdot r + \beta \cdot \phi$ with a partition of unity $\phi$ coming from a covering of $X$ of multiplicity at most $m + 1$.

**Proposition 3.3.** *Suppose $X$ is metric, $m \geq 0$, $t > 0$, and $r_2 > r_1 > 0$. There is $s > 0$ such that if any finite $r$-Lebesque cover $\mathcal{U} = \{U_0, \ldots, U_{m+1}\}$ of $X$, where $r_1 < r < r_2$, admits an $t \cdot r$-Lebesque refinement $\mathcal{V}$ satisfying $m(\mathcal{V}) \leq m + 1$, then any $\lambda$-Lipschitz function $f : A \to S^m$, $A \subset X$, admits a $C \cdot \lambda$-Lipschitz extension $\tilde{f} : X \to S^m$ provided $\frac{1}{12sr_2(m+2)} < \lambda < \frac{1}{12sr_1(m+2)}$ and $C = 50(m+2)^2 s + \frac{150s^2(m+2)^5}{t}$.*

**Proof.** $s > 0$ is chosen so that given a $\lambda$-Lipschitz $f : A \to \Delta^{m+1}$ one can extend it to an $s \cdot \lambda$-Lipschitz $g : X \to \Delta^{m+1}$ (see 3.2).

Suppose $\frac{1}{12sr_2(m+2)} < \lambda < \frac{1}{12sr_1(m+2)}$ and $f : A \to \partial\Delta^{m+1}$ is $\lambda$-Lipschitz. Extend it to an $s \cdot \lambda$-Lipschitz $g : X \to \Delta^{m+1}$. Let $\alpha : X \to [0,1]$ be defined as $\alpha(x) = (m+2) \cdot \min\{g_i(x) \mid 0 \leq i \leq m+1\}$. Notice $Lip(\alpha) \leq (m+2)s \cdot \lambda$. Let $\beta : [0,1] \to [0,1]$ be defined by $\beta(z) = 3z - 1$ on $[1/3, 2/3]$, $\beta(z) = 0$ for $z \leq 1/3$ and $\beta(z) = 1$ for $z \geq 2/3$. Notice $Lip(\beta) \leq 3$.

Put $U_i = \{x \in X \mid g_i(x) > \frac{\alpha(x)}{m+2}$ or $\alpha(x) > 2/3\}$ and notice $L(\mathcal{U}) \geq r = \frac{1}{12s\lambda(m+2)}$ as follows:

Case 1: $x \in X$ and $\alpha(x) > 3/4$. Now, for any $y \in X$ with $d(x,y) < \frac{1}{12s\lambda(m+2)}$ one has $\alpha(x) - \alpha(y) \leq 1/12$, so $\alpha(y) \leq 2/3$ is not possible. Thus, in that case the ball $B(x, \frac{1}{12s\lambda(m+2)})$ is contained in all $U_i$.

Case 2: $\alpha(x) \leq 3/4$. There is $i$ so that $g_i(x) \geq \frac{1}{m+2}$. Since $\psi_i = g_i - \frac{\alpha}{m+2}$ is $2s\lambda$-Lipschitz, for any $y \in X$ satisfying $d(x,y) < \frac{1}{8s\lambda(m+2)}$ one has $\psi_i(x) - \psi_i(y) < \frac{1}{4(m+2)}$ and $\psi_i(y) > 0$ as $\psi_i(x) \geq \frac{1}{4(m+2)}$.

Thus $\mathcal{U}$ is $r$-Lebesque and $r_1 < r < r_2$. Shrink each $U_i$ to $V_i$ so that $m(\mathcal{V}) \leq m + 1$ and $L(\mathcal{V}) \geq \frac{t}{12s\lambda(m+2)}$. The barycentric map $\phi : X \to \partial\Delta^{m+1}$ corresponding to $\mathcal{V}$ has $Lip(\phi) \leq \frac{4(m+2)^2}{L(\mathcal{V})} \leq \frac{48s\lambda(m+2)^3}{t}$.

Define $h(x) = \sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)} \cdot e_i + \sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i$. To show $Lip(h) \leq C \cdot \lambda$ we will use the following observations.

(1) If $u, v : X \to [0, M]$, then $Lip(u \cdot v) \leq M \cdot (Lip(u) + Lip(v))$.
(2) In addition, if $v : X \to [k, M]$ and $k > 0$, then
$$Lip(\frac{u}{v}) \leq M \cdot \frac{Lip(u) + Lip(v)}{k^2}.$$
(3) $v(x) = 1 - \alpha(x) \geq 1/3$ if $\frac{1-\beta(\alpha(x))}{1-\alpha(x)} > 0$.

Therefore $Lip(\sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i) \leq 3 \cdot Lip(\alpha) \cdot (m+2) \cdot Lip(\phi) \leq 3(m+2) \cdot (m+2)s \cdot \lambda \cdot \frac{48s\lambda(m+2)^3}{t} \leq \frac{150s^2\lambda(m+2)^5}{t}$. Also, $Lip(\frac{1-\beta(\alpha(x))}{1-\alpha(x)}) \leq 9 \cdot 4 \cdot Lip(\alpha) \leq 36(m+2)s\lambda$, so $Lip(\sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)}) \leq (m+2) \cdot (2s\lambda + 36(m+2)s\lambda) \leq 50(m+2)^2 s\lambda$ and $C = 50(m+2)^2 s + \frac{150s^2(m+2)^5}{t}$ works.

It remains to show $h(X) \subset \partial\Delta^{m+1}$ and $h|A = f$. $h|A = f$ follows from the fact $\alpha(x) = 0$ if $x \in A$. It is clear $h(x) \in \partial\Delta^{m+1}$ if either $\beta(\alpha(x)) = 0$ or $\beta(\alpha(x)) = 1$, so assume $0 < \beta(\alpha(x)) < 1$. In that case $\phi_i(x) > 0$ implies $g_i(x) - \frac{\alpha(x)}{m+2} > 0$, so the only possibility for $h(x)$ to miss $\partial\Delta^{m+1}$ is when $g_i(x) - \frac{\alpha(x)}{m+2} > 0$ for all $i$ which is not possible. ∎

Propositions 3.1 and 3.4 imply the following

**Corollary 3.4.** *If $X$ is metric and $m \geq 0$, then the following conditions are equivalent:*

   a. *$S^m$ is a Lipschitz extensor of $X$.*
   b. *There is $t > 0$ such that any finite cover $\mathcal{U} = \{U_0, \ldots, U_{m+1}\}$ of $X$ admits a refinement $\mathcal{V}$ so that $L(\mathcal{V}) \geq tL(\mathcal{U})$ and the multiplicity of $\mathcal{V}$ is at most $m + 1$.*

**Proposition 3.5.** *Suppose $X$ is a metric space, $n \geq 0$, $1 > t > 0$, and $r_2 > r_1 > 0$. If every $r$-Lebesque cover $\mathcal{U} = \{U_0, \ldots, U_{n+1}\}$ of $X$, where $r_1 < r < r_2$, admits a $4t \cdot r$-Lebesque refinement $\mathcal{V}$ satisfying $m(\mathcal{V}) \leq n+1$, then any $s$-Lebesque cover $\mathcal{W} = \{W_0, \ldots, W_{n+2}\}$ of $X$, where $4r_1 < s < 4r_2$, admits a $t \cdot s$-Lebesque refinement $\mathcal{V}$ of multiplicity at most $n + 2$.*

**Proof.** Suppose $4r_1 < s < 4r_2$. First, let us show that any $s/2$-Lebesque cover $\mathcal{U} = \{U_i\}_{i=0}^{i=n+1}$ of $A \subset X$ consisting of $n+2$ elements has a refinement $\mathcal{V}$ such that $L(\mathcal{V}) \geq t \cdot s$ and $m(\mathcal{V}) \leq n+1$. Define $U_i' = U_i \cup (X \setminus A)$ for $i \leq n+1$. Notice $L(\mathcal{U}') \geq s/4$. Indeed, if $x \in X$, then $B(x, s/4) \cap A$ is either empty or is contained in $B(y, s/2)$ for some $y \in A$. Since $B(y, s/2) \cap A \subset U_i$ for some $i \leq n+1$, $B(y, s/2) \subset U_i'$ and $B(x, s/4) \subset U_i'$. There is a cover $\mathcal{W}$ of $X$ such that $\mathcal{W}$ refines $\mathcal{U}'$, $L(\mathcal{W}) \geq 4t \cdot s/4$, and $m(\mathcal{W}) \leq n+1$. Putting $\mathcal{V} = \mathcal{W}|_A$ finishes the task.

Suppose $\mathcal{W} = \{W_0, \ldots, W_{n+2}\}$ is an $s$-Lebesque cover of $X$. Let $A$ be the union of balls $B(x, s/2)$ such that $B(x, s)$ is not contained in $W_{n+2}$. Define $U_i = W_i \cap A$ for $i \leq n+1$ and observe $L(\mathcal{U}) \geq s/2$ for $\mathcal{U} = \{U_i\}_{i=0}^{i=n+1}$ as a cover of $A$. Indeed, if $x \in A$, then there is $y \in X$ such that $B(y, s)$ is not contained in $W_{n+2}$ and $x \in B(y, s/2)$. Therefore, $B(y, s) \subset W_i$ for some $i \leq n+1$ which means $B(x, s/2) \cap A \subset B(y, s) \cap A \subset W_i \cap A = U_i$.

Shrink each $U_i$ to $V_i$ so that the intersection of all $V_i$ is empty and $L(\mathcal{V}) \geq t \cdot s$. Define $W_i' = V_i$ for $i \leq n+1$ and $W_{n+2}' = W_{n+2}$. The cover $\mathcal{W}'$ is of multiplicity at most $n+2$. We want to show $L(\mathcal{W}') \geq t \cdot s$. If $B(x, s) \subset W_{n+2}$, we are done. Otherwise $B(x, s/2) \subset A$ and there is $i \leq n+1$ such that $B(x, t \cdot s) \subset V_i$ in which case $B(x, t \cdot s) \subset W_i'$. ∎

**Corollary 3.6.** *Suppose $X$ is a metric space and $n \geq 0$. If $S^n$ is a Lipschitz extensor of $X$, then so is $S^{n+1}$.*

**Proof.** By 3.4 there is $t > 0$ such that any cover $\mathcal{U}$ of $X$ consisting of $n+2$ elements has a refinement $\mathcal{V}$ satisfying $L(\mathcal{V}) \geq t \cdot L(\mathcal{U})$ and $m(\mathcal{V}) \leq n+1$. Use $\lambda_2$ very large and $\lambda_1$ very small. We may assume $t < 1$, so applying 3.5 and 3.4 completes the proof. ∎

**Definition 3.7.** A metric space $E$ is a *large scale Lipschitz extensor* (respectively, *a small scale Lipschitz extensor*) of $X$ if there are constants $C, M > 0$ such that any $\lambda$-Lipschitz function $f : A \to E$, $A$ a subset of $X$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to E$ for all $\lambda < M$ (respectively, for all $\lambda > M$).

Using 3.1, 3.3, and 3.5 one proves easily the following large/small scale analogs of 3.4.

**Corollary 3.8.** *If $X$ is metric and $m \geq 0$, then the following conditions are equivalent:*

    a. *$S^m$ is a large scale Lipschitz extensor (respectively, a small scale Lipschitz extensor) of $X$.*

    b. *There are constants $t, M > 0$ such that any finite $r$-Lebesque cover $\mathcal{U} = \{U_0, \dots, U_{m+1}\}$ of $X$, where $r > M$ (respectively, $r < M$), admits a $t \cdot r$-Lebesque refinement $\mathcal{V}$ of multiplicity at most $m + 1$.*

As in 3.6 one proves its large/small scale analogs:

**Corollary 3.9.** *Suppose $X$ is a metric space and $n \geq 0$. If $S^n$ is a large scale Lipschitz extensor (respectively, a small scale Lipschitz extensor) of $X$, then so is $S^{n+1}$.*

## 4. Lipschitz extensions and Nagata-Assouad dimension

**Theorem 4.1.** *Suppose $X$ is a metric space of finite Nagata-Assouad dimension. If $n \geq 0$, then the following conditions are equivalent:*

    a. *$S^n$ is a Lipschitz extensor of $X$.*

    b. *$dim_{NA}(X) \leq n$.*

**Proof.** The direction (b) $\implies$ (a) follows from 3.4 as follows. Given a cover $\mathcal{U}$ of $X$ of Lebesque number $L(\mathcal{U}) > 0$ pick $\mathcal{V}$ of mesh$(\mathcal{V}) < L(\mathcal{U})$ so that $L(\mathcal{V}) > L(\mathcal{U})/C$ and $m(\mathcal{V}) \leq m + 1$. Notice that $\mathcal{V}$ refines $\mathcal{U}$ and $L(\mathcal{V}) > L(\mathcal{U})/C$.

(a) $\implies$ (b). Without loss of generality (in view of 3.6), we may assume $\dim_{NA}(X) \leq n + 1$. $k > 0$ is chosen so that given a $\lambda$-Lipschitz $f : A \to \partial\Delta^{n+1}$ one can extend it to an $k \cdot \lambda$-Lipschitz $g : X \to \partial\Delta^{n+1}$. Let $c > 1$ be a constant such that for any $r > 0$ there is a cover $\mathcal{U}$ of $X$ of mesh at most $c \cdot r$, Lebesque number at least $r$, that can be expressed as a union $\bigcup_{i=1}^{n+2} \mathcal{U}_i$ so that each $\mathcal{U}_i$ is $r$-disjoint. For such a cover pick a barycentric map $f : X \to \mathcal{N}(\mathcal{U})$ to the nerve of $\mathcal{U}$ such that $Lip(f) \leq \frac{4(n+2)^2}{r}$. Given an $(n+1)$-simplex $\Delta$ in $\mathcal{N}(\mathcal{U})$ we look at $f|f^{-1}(\partial\Delta)$ and extend it over $f^{-1}(\Delta)$ to obtain $g_\Delta : f^{-1}(\Delta) \to \partial\Delta$ of Lipschitz number at most $k \cdot \frac{4(n+2)^2}{r}$. Paste all $g_\Delta$ together to $g : X \to \mathcal{N}(\mathcal{U})$. Our goal it to estimate the mesh and Lebesque number of $g^{-1}(st(v))$, $v$ a vertex of $\mathcal{N}(\mathcal{U})$.

The mesh of $\{g^{-1}(st(v))\}$ is at most twice that of $\{f^{-1}(st(v))\}$. Indeed, if $g(x) \in st(w)$ and $g(x) \neq f(x)$, then $f(x)$ must belong to the interior of a

simplex $\Delta$ containing $w$. Thus $g(x)$ belongs to the star of $st(w)$ in $\{st(v)\}$ and $x$ belongs to the star of $U_w$ in $\mathcal{U}$ ($U_w$ is the element of $\mathcal{U}$ corresponding to $w$). That star is of size at most $4cr$.

Suppose $C$ is a subset of $X$ of diameter less than $\frac{r}{4k\cdot(n+3)^3}$. Pick all elements $U_0, \dots, U_m$ of $\mathcal{U}$ intersecting $C$ and let $v_i$ be corresponding vertices of $\mathcal{N}(\mathcal{U})$. Notice $m \leq n+1$ and $g(C)$ is contained in the simplex $[v_0, \dots, v_m]$ of $\mathcal{N}(\mathcal{U})$. Pick $x_0 \in C$ and, without loss of generality, assume the barycentric coordinate $\phi_{v_0}(x_0)$ of $g(x_0)$ corresponding to $v_0$ is at least $\frac{1}{n+2}$. Suppose $g(x)$ does not belong to $st(v_0)$ for some $x \in C$. Thus $\phi_{v_0}(x) = 0$ and
$$\frac{1}{n+2} \leq |\phi_{v_0}(x) - \phi_{v_0}(x_0)| \leq (k\cdot\tfrac{(n+3)^2}{r})\cdot d(x, x_0) \leq (k\cdot\tfrac{4(n+3)^2}{r})\cdot\tfrac{r}{4k\cdot(n+3)^3} = \tfrac{1}{n+3},$$
a contradiction.

By setting $d = \frac{r}{4k\cdot(n+3)^3}$ the argument above shows the existence of a cover $\mathcal{V}$ of $X$ of multiplicity at most $n+1$, of Lebesque number at least $d$, and of mesh at most $d \cdot 16c \cdot k \cdot (n+3)^3$. That means $\dim_{NA}(X) \leq n$. $\blacksquare$

Adjusting the proof of 4.1 one can deduce the following.

**Theorem 4.2.** *Suppose $X$ is a metric space of finite capacity dimension (respectively, of finite asymptotic dimension with Higson property). If $n \geq 0$, then the following conditions are equivalent:*

    a. *$S^n$ is a small scale (respectively, large scale) Lipschitz extensor of $X$.*

    b. *The capacity dimension (respectively, the asymptotic dimension with Higson property) of $X$ is at most $n$.*

**Problem 4.3.** Suppose $X$ is a metric space such that $S^n$ is a Lipschitz extensor of $X$. Is $\dim_{NA}(X)$ at most $n$?

In [6] the authors proved the following.

**Theorem 4.4.** *For a metric space $X$ the following conditions are equivalent:*

    a. *$\dim_{NA}(X) \leq 0$.*

    b. *Every metric space $Y$ is a Lipschitz extensor of $X$.*

    c. *The 0-sphere $S^0$ is a Lipschitz extensor of $X$.*

Thus, for $n = 0$, the answer to 4.3 is positive.

In the remainder of this section we extend 4.4 to large and small scales.

**Corollary 4.5.** *For a metric space $X$ the following conditions are equivalent:*

    a. *The capacity dimension of $X$ is at most $0$.*

    b. *Each bounded metric space $Y$ is a small scale Lipschitz extensor of $X$.*

    c. *The 0-sphere $S^0$ is a small scale Lipschitz extensor of $X$.*

**Proof.** a) $\implies$ b). Pick $\epsilon > 0$. Let $d = \min(d_X, \epsilon)$ and let $Y$ be $S$-bounded. Define the relation between $M$ and $\epsilon$ by $M = \frac{S}{\epsilon}$ and notice that for $\lambda \geq M = \frac{S}{\epsilon}$, a function $f : (A, d_X|A) \to (Y, d_Y)$ is $\lambda$-Lipschitz if and only if the function $f : (A, d|A) \to Y$ is $\lambda$-Lipschitz.

Let $C > 1$ be a constant such that any $\lambda$-Lipschitz $f : (A, d|A) \to (Y, d_Y)$ extends to $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \to (Y, d_Y)$. Given $\lambda$-Lipschitz function $f : (A, d_X|A) \to (Y, d_Y)$, $\lambda \geq M = \frac{S}{\epsilon}$, the function $f : (A, d|A) \to Y$ is also $\lambda$-Lipschitz. Extend it to a $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \to Y$ and notice that $\tilde{f} : (X, d_X) \to Y$ is also $C \cdot \lambda$-Lipschitz.

b) $\Longrightarrow$ c) is obvious.

c) $\Longrightarrow$ a). Let $M > 1$ be a number such that any $f : A \to S^0$ satisfying $Lip(f) \geq M$ has an extension $g : X \to S^0$ so that $Lip(g) \leq C \cdot Lip(f)$, where $C > 1$. Suppose $r < \frac{1}{(C+1)M}$. Consider the equivalence classes determined by $x \sim y$ if and only if $x$ can be connected to $y$ by a chain of points separated by at most $r$. If any of them has diameter bigger that $Cr$, then there are points $x$ and $y$ in that particular class such that $Cr \leq d(x, y) \leq (C + 1)r$. Pick injection $f : \{x, y\} \to S^0$. Its Lipschitz constant is at least $\frac{1}{Cr+r} > M$ and at most $\frac{1}{Cr}$. By extending it to $g : B \to S^0$ of Lipschitz constant at most $\frac{1}{r}$ we arrive at a contradiction that points of the chain joining $x$ and $y$ are mapped to the same point by $g$. By 2.13 the capacity dimension of $X$ is at most 0. ∎

**Corollary 4.6.** *For a metric space $X$ the following conditions are equivalent:*

a. *The asymptotic dimension of $X$ is at most $0$ with the Higson property.*
b. *Each discrete metric space $Y$ is a large scale Lipschitz extensor of $X$.*
c. *The $0$-sphere $S^0$ is a large scale Lipschitz extensor of $X$.*

**Proof.** a) $\Longrightarrow$ b). Pick $\epsilon > 0$. Let $d = \max(d_X, \epsilon)$ and let $Y$ be $\delta$-discrete. Define the relation between $M$ and $\epsilon$ by $M = \frac{\delta}{\epsilon}$ and notice that for $\lambda \leq M = \frac{\delta}{\epsilon}$, a function $f : (A, d_X|A) \to (Y, d_Y)$ is $\lambda$-Lipschitz if and only if the function $f : (A, d|A) \to Y$ is $\lambda$-Lipschitz.

Let $C > 1$ be a constant such that any $\lambda$-Lipschitz $f : (A, d|A) \to (Y, d_Y)$ extends to $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \to (Y, d_Y)$. Given $\lambda$-Lipschitz function $f : (A, d_X|A) \to (Y, d_Y)$, $\lambda \leq M = \frac{\delta}{C\epsilon}$, the function $f : (A, d|A) \to Y$ is also $\lambda$-Lipschitz. Extend it to a $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \to Y$ and notice that $\tilde{f} : (X, d_X) \to Y$ is also $C \cdot \lambda$-Lipschitz.

b) $\Longrightarrow$ c) is obvious.

c) $\Longrightarrow$ a). Let $M > 0$ be a number such that any $f : A \to S^0$ satisfying $Lip(f) \leq M$ has an extension $g : X \to S^0$ so that $Lip(g) \leq C \cdot Lip(f)$, where $C > 1$. Suppose $r > \frac{1}{CM}$. Consider the equivalence classes determined by $x \sim y$ if and only if $x$ can be connected to $y$ by a chain of points separated by less than $r$. If any of them has diameter bigger that $Cr$, then there are points $x$ and $y$ in that particular class such that $Cr > d(x, y)$. Pick injection $f : \{x, y\} \to S^0$. Its Lipschitz constant is less than $\frac{1}{Cr} < M$. By extending it to $g : B \to S^0$ of Lipschitz constant less than $\frac{1}{r}$ we arrive at a contradiction

that points of the chain joining $x$ and $y$ are mapped to the same point by $g$. By 2.8 the asymptotic dimension of $X$ with Higson property is at most 0. ∎

A way to probe solving 4.3 would be to investigate, for a given $n > 0$, the class of metric spaces $X$ such that $S^n$ is a Lipschitz extensor of $X$. One faces immediately the question of extending 2.15:

**Problem 4.7.** Suppose $X = A \cup B$ is a metric space such that $S^n$ is a Lipschitz extensor of $A$ and $B$. Is $S^n$ is a Lipschitz extensor of $X$?

## 5. Coarsely equivalent metrics and Nagata-Assouad dimension

In this section we characterize asymptotic dimension of Gromov in terms of Nagata-Assouad dimension.

**Theorem 5.1.** *For an unbounded metric space $(X, d)$ the following conditions are equivalent:*

  a. *$asdim(X) \leq n$.*
  b. *There is a hyperbolic metric $(X, d_h)$ coarsely equivalent to $(X, d)$ such that $dim_{NA}(X, d_h) \leq n$ and the Gromov boundary $\partial_\infty X$ of $X$ consists of one point.*
  c. *There is a metric $(X, d_1)$ coarsely equivalent to $(X, d)$ such that Nagata-Assouad dimension $dim_{NA}(X, d_1)$ of $(X, d_1)$ is at most $n$.*

**Proof.** a) $\implies$ b). Pick a sequence of covers $\mathcal{U}_i$ of $X$, $i \geq 1$, of multiplicity at most $n + 1$ such that $mesh(\mathcal{U}_i) \to \infty$, $L(\mathcal{U}_i) \to \infty$, and $2mesh(\mathcal{U}_i) < L(\mathcal{U}_{i+1})$ for all $i$. If $x \neq y$, define $d_h(x, y)$ as the smallest $i$ so that there is $U \in \mathcal{U}_i$ containing both $x$ and $y$. Clearly, $d_h$ is coarsely equivalent to $d$: if $d_h(x, y) \leq i$, then $d(x, y) \leq mesh(\mathcal{U}_i)$. Also, $d(x, y) \leq L(\mathcal{U}_i)$ implies $d_h(x, y) \leq i$.

Notice that for any triangle in $(X, d_h)$ with sides $a \geq b \geq c$ one has $a \leq b + 1$. The reason for this is that $x, y \in U \in \mathcal{U}_i$ and $y, z \in V \in \mathcal{U}_i$ implies existence of $W \in \mathcal{U}_{i+1}$ containing all three points $x, y, z$ as $U \cup V$ is of diameter less than the Lebesque number of $\mathcal{U}_{i+1}$. Thus, in any triangle of $(X, d_h)$ the difference of any two sides that are not minimal is either $-1$, 0, or 1.

Fix $x_0 \in X$ and consider the Gromov product

$$(x|y) = \frac{d_h(x, x_0) + d_h(y, x_0) - d_h(x, y)}{2}.$$

To show $(X, d_h)$ is Gromov hyperbolic it suffices to prove

$$(x|z) \geq \min((x|y), (y|z)) - 4$$

for all $x, y, z \in X$. Equivalently, the smallest product in a triangle is at least the medium one minus 4.

If all distances $d_h(x, x_0)$, $d_h(y, x_0)$, and $d_h(z, x_0)$ are within 1.5 from a number $t$, then, as $d_h(z, x)$ cannot be larger than both $d_h(y, x) + 1$ and

$d_h(y,z) + 1$, we may assume $d_h(z,x) \leq d_h(y,x) + 1$. Now $(x|z) \geq (t - 1.5 + t - 1.5 - (d_h(y,x) + 1))/2 = ((t + 1.5) + (t + 1.5) - d_h(y,x))/2 - 3.5 \geq (x|y) - 4$.

Arrange points $x$, $y$, and $z$ as $u$, $v$, and $w$ so that $s = d_h(u,x_0) \leq m = d_h(v,x_0) \leq l = d_h(w,x_0)$. We may assume $s \leq l - 4$ (otherwise $s$, $m$, and $l$ lie within 1.5 from $(s+l)/2$).

Case 1: $m \leq l - 2$. Now $d_h(w,v)$ must be within 1 from $l$, so $2(w|v) = m + l - d_h(w,v)$ is contained between $m - 1$ and $m + 1$. Similarly, $s - 1 \leq 2(u|w) \leq s + 1$. Since $d_h(u,v) \leq m + 1$, $2(u|v) \geq m + s - (m+1) = s - 1$. The only possibility for the smallest of Gromov products for the triple $uvw$ to be less than the medium one minus 4 is if $s \leq m - 2$. In that case $d_h(u,v) \geq m - 1$, so $s + 1 = m + s - (m-1) \geq 2(u|v) \geq m + s - (m+1) = s - 1$ and the smallest of Gromov products for the triple $uvw$ is larger than the medium one minus 4.

Case 2: $m \geq l - 1$. Now $d_h(u,v)$ must be within 1 from $m$, so $2(u|v) = m + s - d_h(u,v)$ is contained between $s - 1$ and $s + 1$. Similarly, $s - 1 \leq 2(u|w) \leq s + 1$. Since $d_h(w,v) \leq l + 1$, $2(w|v) \geq m + l - (l+1) = m - 1 \geq s + 2$, and the smallest of Gromov products for the triple $uvw$ is larger than the medium one minus 4.

To prove $\dim_{NA}(X, d_h) \leq n$ we plan to define covers $\mathcal{V}_r$ of $(X, d_h)$ such that $\text{mesh}(\mathcal{V}_r) \leq r$, $L(\mathcal{V}_r) \geq r/4$, and $m(\mathcal{V}_r) \leq n + 1$. If $r \leq 4$, we define $\mathcal{V}_r$ as all singletons of $X$, otherwise we put $\mathcal{V}_r = \mathcal{U}_i$ with $i$ being the integral part of $r$. Indeed, for $r > 4$, $\text{mesh}(\mathcal{U}_i) \leq i \leq r$ and $L(\mathcal{U}_i) \geq i - 1 \geq r - 2 \geq r/4$.

Let us show the Gromov boundary $\partial_\infty X$ of $X$ consists of one point. Given two sequences of points $\{x_i\}$ and $\{y_i\}$ such that $(x_m|x_k) \to \infty$ and $(y_m|y_k) \to \infty$ we need to prove $(x_m,y_m) \to \infty$. However, $d_h(x_m,y_m)$ is smaller than $\max(d_h(x_m,x_0), d_h(y_m,x_0)) + 2$, so $2(x_m|y_m) \geq \min(d_h(x_m,x_0), d_h(y_m,x_0)) - 2 \to \infty$.

Since b) $\Longrightarrow$ c) and c) $\Longrightarrow$ a) are obvious, we are done. ∎

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